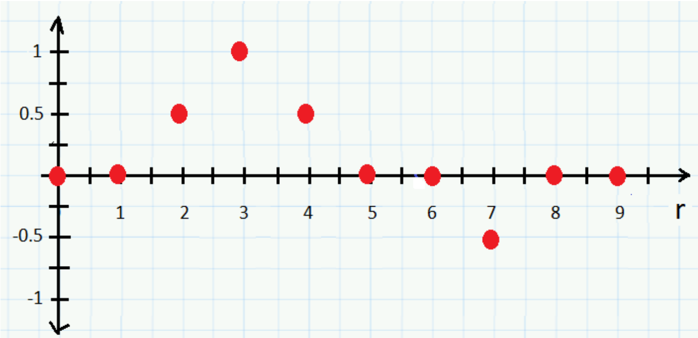
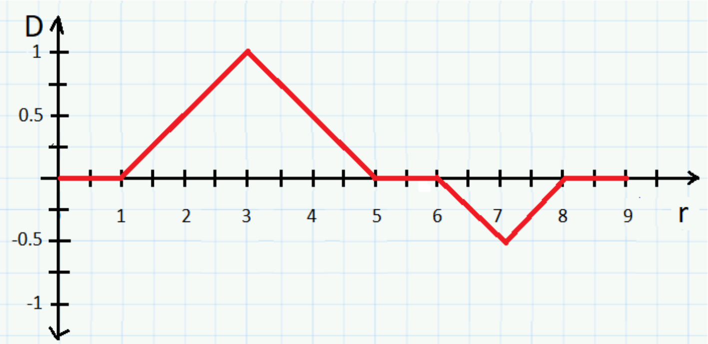
B.2 Description of wave (3D)

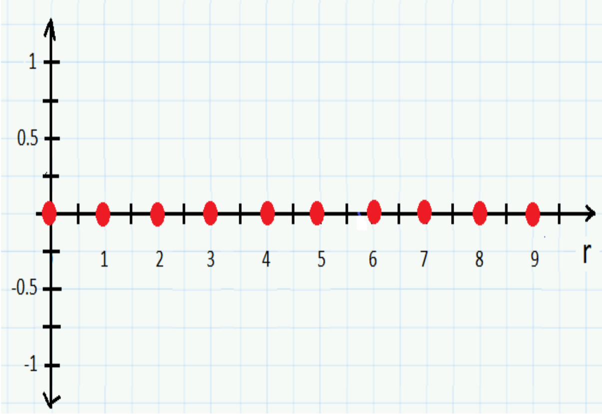
Transverse displacement
(perpendicular to radial direction)



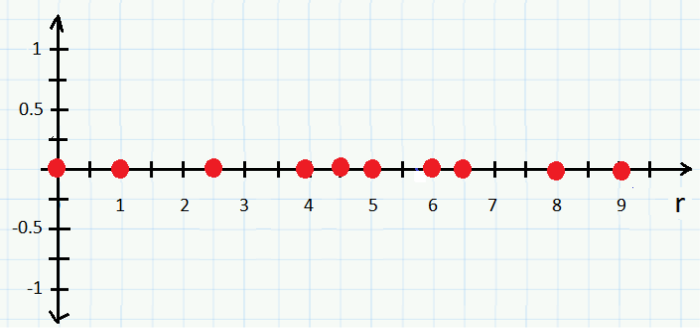
Graphical/Mathematical description



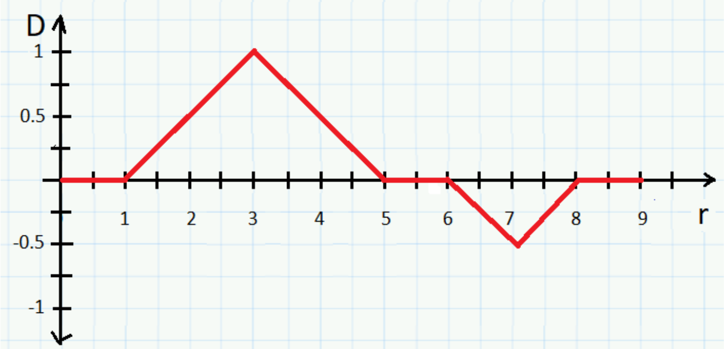
Equilibrium Position



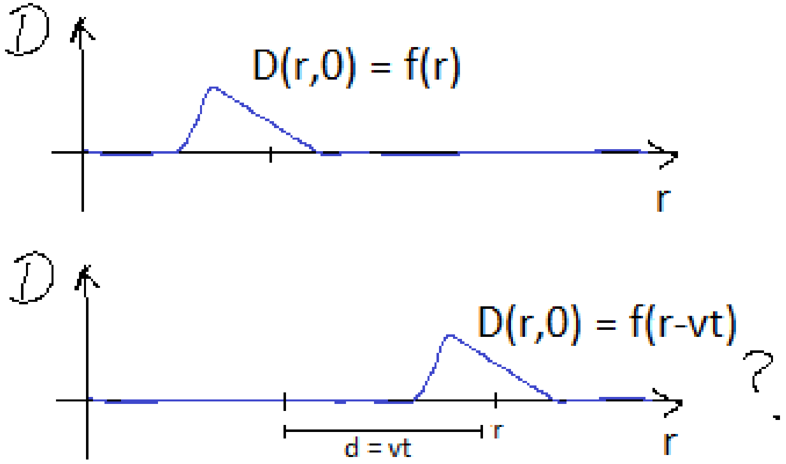
Longitudinal displacement
(parallel to radial direction)



Graphical/Mathematical description



B.2 How does displacement propagate through medium (3D)?



Suppose we create an initial disturbance, of shape D(r,0) = f(r).

Would this disturbance propagate radially outwards with same shape, and constant velocity v?

Uh no, actually. To see what *does* happen, we would have to solve N2L.

B.2 N2L Analysis of Wave Propagation (3D)

Consider for example, a wave propagating through a
a medium with mass density ρ, and under ‘tension’ B.

$$\rho = \text{density} = \frac{dm}{dV} \qquad [\text{units} = \text{kg/ m}^3]$$

$$B = \text{bulk modulus} \qquad [\text{units = N/m}^2]$$

~ Pressure under which medium is held together

Generalizing the 1D analysis to 3D, accounting for the fact that the forces can act in all three directions now, And that B takes the place of T, and ρ of μ, we get:

$$B\left(\frac{d^2D}{dx^2}+\frac{d^2D}{dy^2}+\frac{d^2D}{dz^2}\right)=\rho\frac{d^2D}{dt^2} \qquad \text{wave equation in 3D}$$

B.2 Solution of Wave Equation (3D)

$$B\left(\frac{d^2D}{dx^2}+\frac{d^2D}{dy^2}+\frac{d^2D}{dz^2}\right)=\rho\frac{d^2D}{dt^2}$$

So this is our wave equation, and we must solve it, but doing so requires a bit of advanced math. Suffice to say, we would find the following fact. If you're into partial derivatives, then you could, in a few lines, verify this is true.

if $\quad D(r,0)=\dfrac{f(r)}{r}$

then $\quad D(r,t)=\dfrac{f(r-vt)}{r} \qquad v=\sqrt{\dfrac{B}{\rho}}$

An important special case is that of sound. In that case, one can show that

$$v_{sound}=\sqrt{\frac{\gamma RT}{m_{molar}}}$$

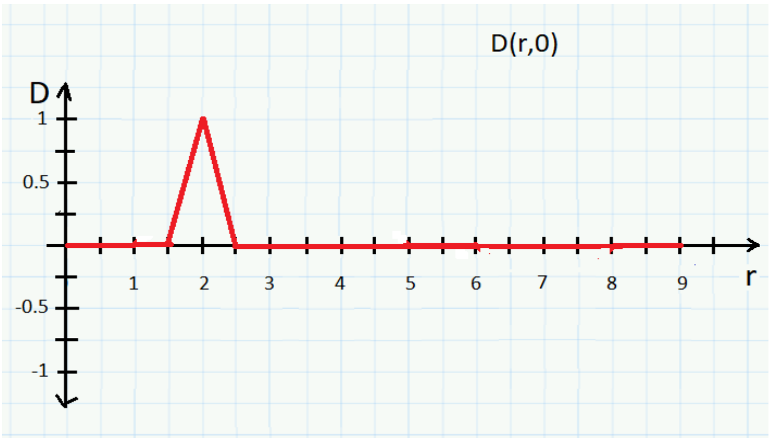
$\gamma=1.4, \quad R=8.31\ \mathrm{J/mol\cdot K},$

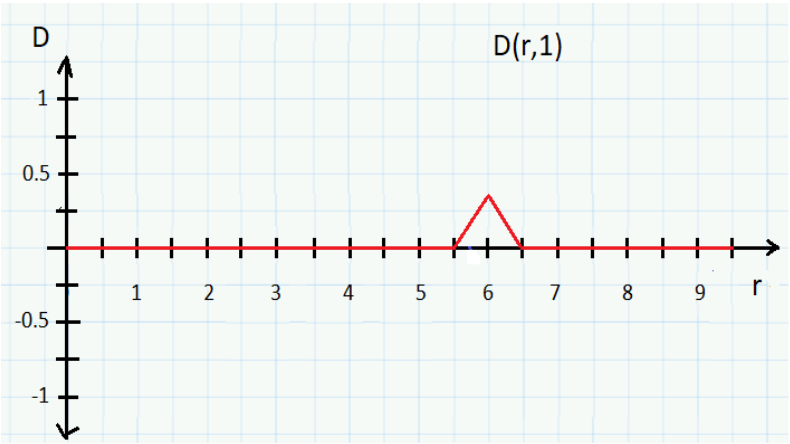
$T=$ temperature of gas in Kelvin

$m_{mol}=$ molar mass (in kg)

B.2 Determining motion of medium from D(r,0) graph/function (3D)

Say initial disturbance D(r,0) is given as graph, and propagates outward at 4m/s. What will it look like 1s later?





Shifts over distance d = vt as usual, But also, amplitude is diminished ~ 1/r (basically, amplitude • radius = constant)

Say initial disturbance D(r,0) is given as function, and propagates outward with speed v. What will it be given by at time t?

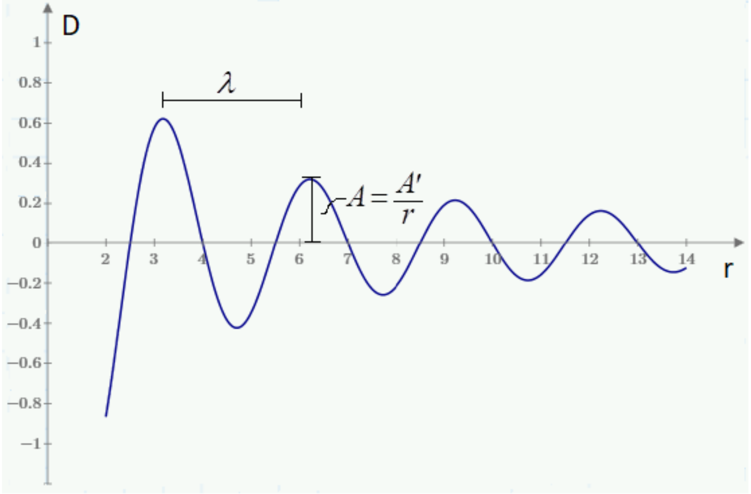
$$D(r,0)=\frac{1}{1+2r^2}$$

$$D(r,t)=\frac{f(r-vt)}{r}$$

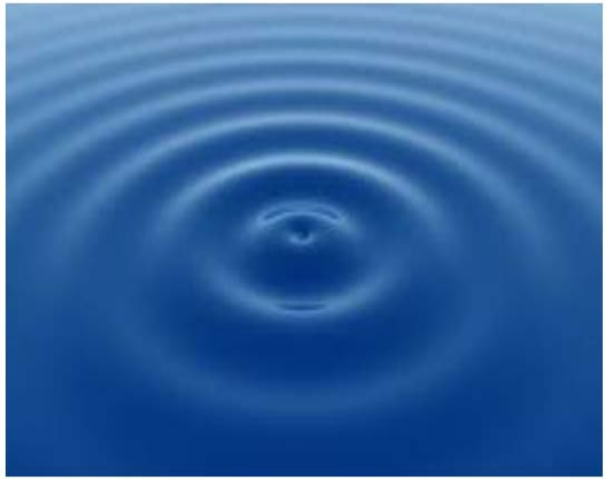
But what is f(r)? Well, $D(r,0)=\frac{f(r)}{r}\rightarrow f(r)=rD(r,0)=\frac{r}{1+2r^2}$ So

$$D(r,t)=\frac{1}{r}\cdot\frac{r-vt}{1+2(r-vt)^2}$$

B.2 Sinusoidal Waves (3D)



Most of the analysis we did on waves in 1D, carries over into 3D. But there is one difference, that you might be familiar with. For instance,if you drop a pebble into a pond, it will produce a sinusoidal ripple - but whose amplitude attenuates with radius. This is the key difference: sinusoidal waves in 3D have a decaying amplitude.



$$D(r,0) = \frac{f(r)}{r} = \frac{A'}{r}\sin(kr + \varphi_0)$$

$$D(r,t) = \frac{f(r - vt)}{r} = \frac{A'}{r}\sin\left[k(r - vt) + \varphi_0\right] = \frac{A'}{r}\sin\left[kr - \omega t + \varphi_0\right]$$

$\frac{A'}{r} = \text{amplitude at radius } r$

$k = \text{'curvature'} = \frac{2\pi}{\lambda}$ $\qquad$ $\lambda = \text{wavelength}$

$\varphi_0 = \text{'phase constant'} = \frac{2\pi}{\lambda}\Delta r_0$

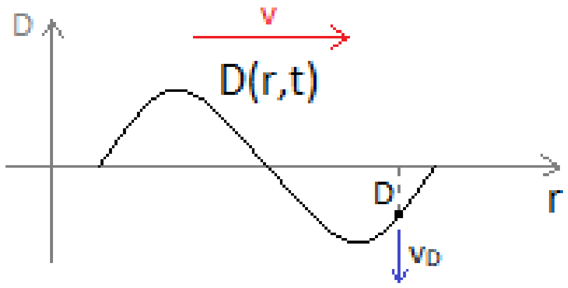
$\omega = \text{'angular frequency'} = \frac{2\pi}{T}$ [units = rad/s]

$T = \text{'period'} = \text{time it takes for a wavelength to pass by}$

$f = \text{'frequency'} = \text{rate at which wavelengths pass by} = \frac{1}{T}$ [units = Hertz (Hz)]

$v = \frac{\lambda}{T} = \lambda f = \omega / k$

B.2 Energy Carried by Sinusoidal Waves (3D)



Just as in 1D, 3D sinusoidal waves impart energy to the particles in the medium through which they travel. And we can calculate what it is in the same way:

$$E_{particle} = \frac{1}{2} m_{particle} v_D^2 + \frac{1}{2} k D^2$$

$k =$ 'spring constant' of the medium

$$= \frac{1}{2} m_{particle} v_D^2 + \frac{1}{2}\left(m_{particle}\omega^2\right) D^2$$

using $\omega = \sqrt{\dfrac{k}{m_{particle}}}$

$$= \frac{1}{2} m_{particle}\left[-\omega \frac{A'}{r}\cos(kr - \omega t + \varphi_0)\right]^2 + \frac{1}{2} m_{particle}\omega^2\left[\frac{A'}{r}\sin(kr - \omega t + \varphi_0)\right]^2$$

$$= \frac{1}{2} m_{particle} \frac{A'^2}{r^2}\omega^2$$

$$E_{volume,dV} = \frac{1}{2}(\rho dV)\left(\frac{A'}{r}\omega\right)^2$$

$dm = \rho dV =$ mass of volume dV

$$= \frac{1}{2}\rho\left(\frac{A'}{r}\omega\right)^2 dV$$

$$E_{volume,V} = \int u dV$$

$$u = \text{'energy density'} = \frac{1}{2}\rho\left(\frac{A'}{r}\omega\right)^2 \quad [\text{units} = \text{J/m}^3]$$

$$I = \text{'intensity'} = uv \qquad [\text{units} = \text{J/(m}^2\text{s)} = \text{W/m}^2]$$

Observe that both energy density and intensity fall of as $1/r^2$. This is why lights get dimmer, and sounds quieter, as distance from source increases.

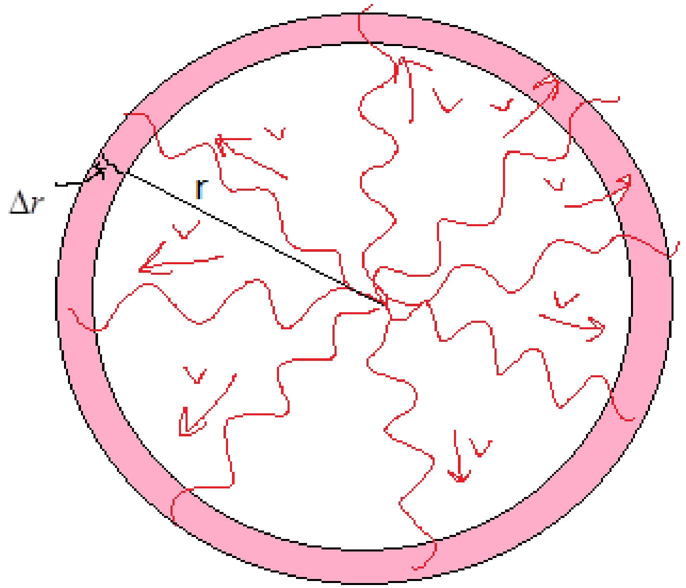
B.2 Energy carried by sinusoidal waves (3D)

Last thing to calculate is power – and that deserves a little extra consideration….

$$P = \frac{\text{energy that flows across surface in time } \Delta t}{\Delta t}$$

$$= \frac{u \cdot (\text{surface area} \cdot \Delta r)}{\Delta t} = uv \cdot \text{surface area} = I \cdot \text{surface area}$$

$$P = I \cdot Area$$



Energy Conservation

An important consequence of the wave equation is energy conservation, i.e., the power that flows across surface of radius r_1 = power that flows across surface of radius r_2. This is evinced by the independence of P from r.

$$\begin{aligned} P &= I \cdot Area \\ &= uv \cdot Area \\ &= \frac{1}{2}\rho\left(\frac{A'}{r}\omega\right)^2 v \cdot 4\pi r^2 \\ &= 2\pi\rho A'^2 \omega v \end{aligned}$$

B.2 Energy carried by sinusoidal waves still, and forever

When a sound wave hits the membrane in your ear, it will make it oscillate back and forth, and the magnitude of these oscillations correlates to the perceived volume of the sound wave. Turns out, due to some biological stuff I don't know, we get the following relationship:

$$\beta = \text{'volume, as in how loud it seems to be'}$$
$$= 10\log\left(\frac{I}{I_0}\right) \qquad \text{[units = decibels (dB)]}$$

$I = \text{sound wave intensity}$

$I_0 = \text{minimum sound wave intensity typical person can hear} = 10^{-12}\ \text{W/m}^2$

$I_{\text{max}} = \text{maximum sound wave intensity typical person can stand} = 1\ \text{W/m}^2$

- The minimum intensity I_0 has a volume β = 0dB, as you can see if you plug it into the equation….
- The maximum intensity I_{max} has a volume β = 120dB, as you can also see….

For example….

A bomb explodes in the air, sending soundwaves propagating outwards according to the following equation (r is measured in meters). The density of air is ρ = 1.2kg/m^3.

$$D(r,t) = \frac{0.01\text{m}^2}{r}\sin(3\text{m}^{-1}\cdot r - 1000\text{rad/s}\cdot t)$$

(a) What are the wavelength, frequency, and speed of the sound wave?

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3} = 2.1\text{m}$$

$$f = \frac{\omega}{2\pi} = \frac{1000}{2\pi} = 159\text{Hz}$$

$$v = f\lambda = 334\,\text{m/s}$$

(b) What is the maximum displacement and maximum speed of an air molecule 5m from the explosion?

$$D_{\max} = D(5,t)\big|_{\max} = \frac{0.01\text{m}^2}{5\text{m}}\sin(3\text{m}^{-1}\cdot 5 - 1000\text{rad/s}\cdot t)\Big|_{\max} = 0.002\text{m}\cdot\sin(15-1000t)\big|_{\max} = 0.002\text{m}$$

$$\text{v}_{\max} = \frac{dD(5,t)}{dt}\Big|_{\max} = -1000\frac{\text{rad}}{\text{s}}\cdot\frac{0.01\text{m}^2}{5\text{m}}\cdot\cos(3\text{m}^{-1}\cdot 5 - 1000\,\text{rad/s}\cdot t)\Big|_{\max} = -2\text{m/s}\cdot\cos(15-1000t)\big|_{\max} = 2\text{m/s}$$

(c) What is the sound wave's power?

$$\begin{aligned} P &= I \cdot Area \\ &= uv \cdot Area \\ &= \frac{1}{2}\rho\left(\frac{A'}{r}\omega\right)^2 (v)(4\pi r^2) \\ &= \frac{1}{2}\rho\left(\frac{A'^2}{r^2}\omega^2\right)(v)\cdot(4\pi r^2) \\ &= 2\pi\rho A'^2\omega^2 v \\ &= 2\pi(1.2)(0.01^2 \times 1000^2)(334) \\ &= 2.52\times 10^5\ \mathrm{W} \end{aligned}$$

(d) How loud is the explosion 200m away?

$$\begin{aligned} \beta &= 10\log\left(\frac{I}{I_0}\right) \\ &= 10\log\left(\frac{0.50}{10^{-12}}\right) \\ &= 117\,\mathrm{dB} \end{aligned}$$

$$I = \frac{P}{4\pi r^2} = \frac{2.52\times 10^5\,\mathrm{W}}{4\pi(200\mathrm{m})^2} = 0.50\,\mathrm{W/m^2}$$